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ELASTIC ANISOTROPIC MATERIAL WITH PURELY LONGITUDINAL AND TRANSVERSE WAVES

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The simplest form of the matrix of elasticity moduli of an anisotropic material conducting purely longitudinal and transverse waves with an arbitrary direction of the wave normal is obtained. A generic solution of equations in displacements is represented in terms of three functions satisfying independent wave equations. In the case of planar deformation, this solution yields a complex representation coinciding with the Kolosov–Muskhelishvili formulas for an isotropic material. The formulas in the present work also determine an anisotropic material with Young's modulus identical for all directions, as in an isotropic medium.

Key words: *anisotropy, longitudinal and transverse waves, moduli of elasticity, generic solution.*

Explicit formulas for anisotropic materials conducting purely longitudinal and transverse waves with an arbitrary direction of the wave normal were derived in [1, 2]; the existence of similar media was later noted in [3–5].

The fourth-rank tensor of Young's moduli $A_{ijkl} = A_{jikl} = A_{klij}$ admits decomposition into a constant (isotropic) term and terms containing two deviators and a nonor and corresponding to irreducible linear representations of the orthogonal group of transformations of the coordinate system x_1, x_2, x_3 [6–9]. The following expansion is proposed in [7, 9]:

$$A_{ijkl} = H(\delta_{ij}\delta_{kl} + 2\delta_{ijkl})/3 + 2h(\delta_{ij}\delta_{kl} - \delta_{ijkl})/3$$

$$+ (H_{ij}\delta_{kl} + H_{kl}\delta_{ij} + H_{ik}\delta_{lj} + H_{lj}\delta_{ik} + H_{il}\delta_{jk} + H_{jk}\delta_{il})/6$$

$$+(h_{ij}\delta_{kl} + h_{kl}\delta_{ij})/3 - (h_{ik}\delta_{lj} + h_{lj}\delta_{ik} + h_{il}\delta_{jk} + h_{jk}\delta_{il})/6 + N_{ijkl}. \qquad (1)$$

Here $\delta_{ijkl} = (\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk})/2$ and $\delta_{ij} = 1$ for $i = j$ or $\delta_{ij} = 0$ for $i \neq j$. The sum of terms with H, H_{ij}, and N_{ijkl} is the symmetric part $A_{(ijkl)} = (A_{ijkl} + A_{iklj} + A_{iljk})/3$, and the sum of terms with h and h_{ij} is the asymmetric part $A_{ijkl} - A_{(ijkl)} = (2A_{ijkl} - A_{iklj} - A_{iljk})/3$. The tensor $N_{ijkl} = N_{(ijkl)}$ is a nonor, and $H_{ij} = H_{(ij)}$ and $h_{ij} = h_{(ij)}$ are deviators, i.e., $N_{iikl} = 0$, $H_{ii} = 0$, and $h_{ii} = 0$. Summation is performed over repeated subscripts, and the subscripts in parentheses denote symmetric functions. All terms of expansion (1) are mutually orthogonal.

The terms A_{ijkl} being defined, all the quantities in the right side of Eq. (1) are uniquely determined, and vice versa, using Eq. (1), one can define A_{ijkl} via two constants H and h, five independent components H_{ij}, five independent components h_{ij}, and nine independent components of the nonor N_{ijkl} [7, 9]. The constant tensor in Eq. (1) is written in the traditional form as

$$H(\delta_{ij}\delta_{kl} + 2\delta_{ijkl})/3 + 2h(\delta_{ij}\delta_{kl} - \delta_{ijkl})/3 = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ijkl},$$

where

$$\lambda = (H + 2h)/3 = (2A_{iikk} - A_{ikki})/15, \qquad h = \lambda - \mu,$$

$$2\mu = 2(H - h)/3 = (3A_{ikki} - A_{iikk})/15, \qquad H = \lambda + 2\mu.$$

The constants λ and μ correspond to the Lamé constants for an isotropic material.

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Using the formulas for passing from two subscripts to one,

$$h_{11} = h_1, \qquad h_{22} = h_2, \qquad h_{33} = h_3,$$

$$\sqrt{2}h_{23} = \sqrt{2}h_{32} = h_4, \qquad \sqrt{2}h_{13} = \sqrt{2}h_{31} = h_5, \qquad \sqrt{2}h_{12} = \sqrt{2}h_{21} = h_6,$$

we write expansion (1) as the sum of the matrices

$$A_{ij} = \frac{H}{3}\begin{bmatrix} 3 & & & & & \\ 1 & 3 & & \text{sym} & & \\ 1 & 1 & 3 & & & \\ 0 & 0 & 0 & 2 & & \\ 0 & 0 & 0 & 0 & 2 & \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} + \frac{2h}{3}\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & \text{sym} & & \\ 1 & 1 & 0 & & & \\ 0 & 0 & 0 & -1 & & \\ 0 & 0 & 0 & 0 & -1 & \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$+ \frac{1}{6}\begin{bmatrix} 6H_1 & & & & & \\ -H_3 & 6H_2 & & & \text{sym} & \\ -H_2 & -H_1 & 6H_3 & & & \\ H_4 & 3H_4 & 3H_4 & -2H_1 & & \\ 3H_5 & H_5 & 3H_5 & \sqrt{2}H_6 & -2H_2 & \\ 3H_6 & 3H_6 & H_6 & \sqrt{2}H_5 & \sqrt{2}H_4 & -2H_3 \end{bmatrix}$$

$$+ \frac{1}{3}\begin{bmatrix} 0 & & & & & \\ -h_3 & 0 & & & \text{sym} & \\ -h_2 & -h_1 & 0 & & & \\ h_4 & 0 & 0 & h_1 & & \\ 0 & h_5 & 0 & -h_6/\sqrt{2} & h_2 & \\ 0 & 0 & h_6 & -h_5/\sqrt{2} & -h_4/\sqrt{2} & h_3 \end{bmatrix}$$

$$+ \begin{bmatrix} N_{11} & & & & & \\ N_{21} & N_{22} & & & \text{sym} & \\ N_{31} & N_{32} & N_{33} & & & \\ N_{41} & N_{42} & N_{43} & 2N_{32} & & \\ N_{51} & N_{52} & N_{53} & \sqrt{2}N_{63} & 2N_{31} & \\ N_{61} & N_{62} & N_{63} & \sqrt{2}N_{52} & \sqrt{2}N_{41} & 2N_{21} \end{bmatrix}. \qquad (2)$$

Here we have $H_1 + H_2 + H_3 = 0$, $h_1 + h_2 + h_3 = 0$, and

$$N_{11} + N_{21} + N_{31} = 0, \qquad N_{41} + N_{42} + N_{43} = 0,$$

$$N_{21} + N_{22} + N_{32} = 0, \qquad N_{51} + N_{52} + N_{53} = 0,$$

$$N_{31} + N_{32} + N_{33} = 0, \qquad N_{61} + N_{62} + N_{63} = 0.$$

The equations of the elasticity theory with arbitrary anisotropy and neglected bulk forces have the following form in the Cartesian rectangular coordinates x_1, x_2, x_3 [1, 2]:

$$L_{ij}u_j = 0, \qquad L_{ij} = L_{ji} = A_{i(kl)j}\partial_{kl} - \rho\delta_{ij}\partial_{44}. \qquad (3)$$

Here u_j is the displacement vectors, $A_{i(kl)j} = (A_{iklj} + A_{ilkj})/2$, ρ is the constant density of the material, ∂_k is the derivative with respect to the coordinate x_k, and ∂_4 is the derivative with respect to time $x_4 = t$.

If differential operators T and $D = \operatorname{diag}(D_1, D_2, D_3)$ with constant coefficients, such that $LT = TD$ ($|T| \neq 0$), are found for operators (3), the generic solution of Eqs. (3) has the form [1, 2]

$$u = T\phi, \qquad D\phi = f, \qquad Tf = 0. \qquad (4)$$

The matrix of elasticity moduli was obtained in [1, 2] (see also [10])

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{11}-A_{66} & A_{11} & & & \text{sym} & \\ A_{11}-A_{55} & A_{11}-A_{44} & A_{11} & & & \\ A_{41} & 0 & 0 & A_{44} & & \\ 0 & A_{52} & 0 & -A_{63}/\sqrt{2} & A_{55} & \\ 0 & 0 & A_{63} & -A_{52}/\sqrt{2} & -A_{41}/\sqrt{2} & A_{66} \end{bmatrix}, \tag{5}$$

where

$$A_{41} = -2\sqrt{2}bc_2c_3, \qquad A_{44} = 2(a - bc_1^2),$$

$$A_{52} = -2\sqrt{2}bc_1c_3, \qquad A_{55} = 2(a - bc_2^2),$$

$$A_{63} = -2\sqrt{2}bc_1c_2, \qquad A_{66} = 2(a - bc_3^2).$$

In this case, the operators T and D are

$$T = [\partial_j,\ \varepsilon_{jmn}c_m\partial_n,\ c_j\partial_{kk} - c_m\partial_{mj}]; \tag{6}$$

$$D_1 = A_{11}\partial_{kk} - \rho\partial_{44}, \quad D_2 = [(a - bc_mc_m)\delta_{kl} + bc_kc_l]\partial_{kl} - \rho\partial_{44}, \quad D_3 = a\partial_{kk} - \rho\partial_{44}. \tag{7}$$

Here ε_{jmn} are the Levi-Civita symbols, A_{11}, a, b, c_1, c_2, and c_3 are arbitrary parameters such that matrix (5) is positively determined, and c_j is a nonzero vector.

By substituting n_k for ∂_k and $|v|^2 = v_iv_i$ ($v_i = |v|n_i$) for ∂_{44}, we find from Eq. (7) the phase velocities corresponding to the wave normal n_k:

$$\begin{gathered} \rho|v|_1^2 = A_{11}n_kn_k = A_{11}, \qquad \rho|v|_3^2 = an_kn_k = a, \\ \rho|v|_2^2 = [(a - bc_mc_m)\delta_{kl} + bc_kc_l]n_kn_l = a - bc_mc_m + bc_kc_ln_kn_l. \end{gathered} \tag{8}$$

It follows from (6) that formulas (5)–(8) define a medium with *purely longitudinal* and *purely transverse waves* [11] with an arbitrary direction of the wave normal n_k. In experiments with longitudinal waves, the material corresponding to matrix (5) cannot be distinguished from an isotropic material [3–5].

Using expansion (1), we find conditions for propagation of purely longitudinal waves in a material. From (1), we obtain

$$A_{ijkl}\partial_{jkl} = (\lambda + 2\mu)\partial_{ikk} + (H_{ik}\partial_{kll} + H_{kl}\partial_{ikl})/2 + N_{ijkl}\partial_{jkl}$$

$$= \partial_i[(\lambda + 2\mu)\partial_{kk} + H_{kl}\partial_{kl}/2] + (H_{ij}\delta_{kl}/2 + N_{ijkl})\partial_{jkl}.$$

It follows from here that, for a purely longitudinal wave to exist, the following condition should be satisfied:

$$(H_{ij}\delta_{kl}/2 + N_{ijkl})\partial_{jkl} = \partial_iB_{kl}\partial_{kl} = \delta_{ij}B_{kl}\partial_{jkl}$$

or

$$(H_{ij}\delta_{kl}/2 + N_{ijkl} - \delta_{ij}B_{kl})\partial_{jkl} = 0. \tag{9}$$

Here $B_{kl} = B_{(kl)}$ are quantities unknown at the moment. Relation (9) is valid if

$$N_{ijkl} = \delta_{i(j}B_{kl)} - H_{i(j}\delta_{kl)}/2 = [\delta_{ij}B_{kl} + \delta_{ik}B_{lj} + \delta_{il}B_{jk} - (H_{ij}\delta_{kl} + H_{ik}\delta_{lj} + H_{il}\delta_{jk})/2]/3. \tag{10}$$

Taking into account the nonor and deviator properties, we obtain from Eq. (10)

$$N_{iikl} = (5B_{kl} - H_{kl})/3 = 0, \qquad B_{kl} = H_{kl}/5,$$

$$N_{ijkl} = [(\delta_{ij}H_{kl} + \delta_{ik}H_{lj} + \delta_{il}H_{jk})/5 - (H_{ij}\delta_{kl} + H_{ik}\delta_{lj} + H_{il}\delta_{jk})/2]/3, \quad N_{ijkk} = -7H_{ij}/10 = 0.$$

Hence, propagation of purely longitudinal waves with an arbitrary direction of the wave normal n_k in an anisotropic material is possible if $H_{ij} = 0$ and $N_{ijkl} = 0$ in Eq. (1).

The deviator h_{ij} does not affect the longitudinal wave. Since h_{ij} is a deviator and (1) [or (2)] is an invariant expansion, we can assume that the coordinate system is the basic one for h_{ij}, i.e., $h_4 = h_5 = h_6 = 0$, and Eq. (2) takes the form

$$A_{ij} = \begin{bmatrix} \lambda+2\mu & & & & & \\ \lambda-h_3/3 & \lambda+2\mu & & & \text{sym} & \\ \lambda-h_2/3 & \lambda-h_1/3 & \lambda+2\mu & & & \\ 0 & 0 & 0 & 2\mu+h_1/3 & & \\ 0 & 0 & 0 & 0 & 2\mu+h_2/3 & \\ 0 & 0 & 0 & 0 & 0 & 2\mu+h_3/3 \end{bmatrix}, \tag{11}$$

$$h_1 + h_2 + h_3 = 0.$$

Expression (11) is the simplest form of the matrix of elasticity moduli of a material in which propagation of purely longitudinal waves with an arbitrary direction of the wave normal n_k is possible.

Comparing (2), (5), (11), for matrix (5) we find

$$\lambda = A_{11} - 2(a - bc_sc_s/3), \quad 2\mu = 2(a - bc_sc_s/3);$$

$$h_{ij} = 2b(c_sc_s\delta_{ij} - 3c_ic_j). \tag{12}$$

It follows from (12) that c_j is an eigenvector of the deviator h_{ij}, and its eigenvalues are $h_1 = h_2 = 2bc_sc_s$ and $h_3 = -4bc_sc_s$. This vector can be considered as a unit vector ($c_sc_s = 1$); in the basic axes, we have $c_j = \delta_{j3}$.

Matrix (5) takes the form

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{11}-2(a-b) & A_{11} & & & \text{sym} & \\ A_{11}-2a & A_{11}-2a & A_{11} & & & \\ 0 & 0 & 0 & 2a & & \\ 0 & 0 & 0 & 0 & 2a & \\ 0 & 0 & 0 & 0 & 0 & 2(a-b) \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & & & & & \\ A_{21} & A_{11} & & & \text{sym} & \\ A_{31} & A_{31} & A_{11} & & & \\ 0 & 0 & 0 & A_{11}-A_{31} & & \\ 0 & 0 & 0 & 0 & A_{11}-A_{31} & \\ 0 & 0 & 0 & 0 & 0 & A_{11}-A_{21} \end{bmatrix}, \tag{13}$$

where $2a = A_{11} - A_{31}$ and $2b = A_{21} - A_{31}$. Obviously, matrix (13) corresponds to a particular case of a transversely isotropic material [11, 12]. The eigen moduli of elasticity $\mu_i > 0$ and the orthogonal eigen states t_{ip} and $t_{ip}t_{iq} = \delta_{pq}$ for Eq. (13) are as follows [12]:

$$\mu_{1,2} = (2A_{11} + A_{21} \pm \sqrt{A_{21}^2 + 8A_{31}^2})/2 = [3A_{11} - 2(a-b) \pm \sqrt{(A_{11} - 2(a-b))^2 + 8(A_{11} - 2a)^2}]/2, \tag{14}$$

$$\mu_3 = \mu_6 = A_{11} - A_{21} = 2(a-b), \qquad \mu_4 = \mu_5 = A_{11} - A_{31} = 2a;$$

$$t_{ip} = \begin{bmatrix} \cos\alpha/\sqrt{2} & -\sin\alpha/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ \cos\alpha/\sqrt{2} & -\sin\alpha/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tan 2\alpha = 2\sqrt{2}A_{31}/A_{21} = 2\sqrt{2}(A_{11} - 2a)/(A_{11} - 2(a-b)).$$

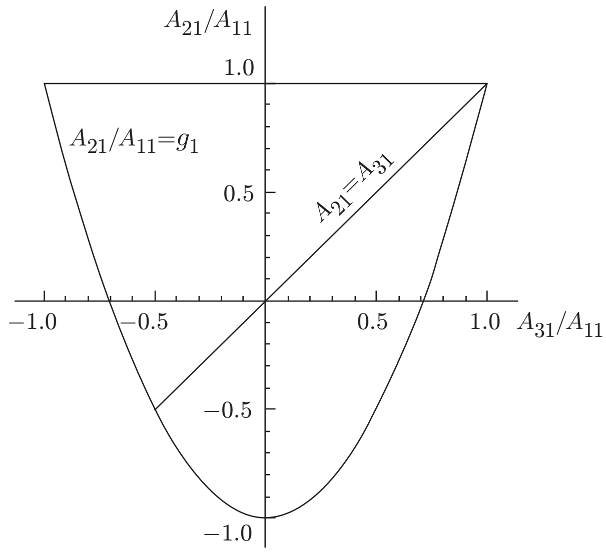

Fig. 1

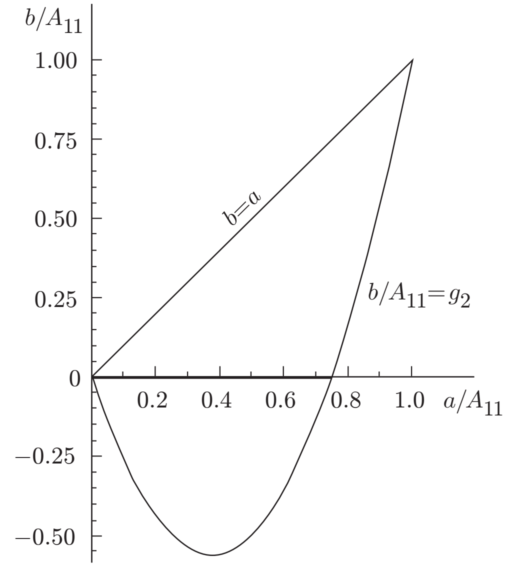

Fig. 2

The expressions for the matrix A_{ij} and the inverse matrix of compliance coefficients $a_{ij} = A_{ij}^{-1}$ in terms of eigen moduli and states have the following form:

$$A_{ij} = \mu_1 t_{i1} t_{j1} + \mu_2 t_{i2} t_{j2} + \mu_3 (t_{i3} t_{j3} + t_{i6} t_{j6}) + \mu_4 (t_{i4} t_{j4} + t_{i5} t_{j5}),$$

$$a_{ij} = t_{i1} t_{j1}/\mu_1 + t_{i2} t_{j2}/\mu_2 + (t_{i3} t_{j3} + t_{i6} t_{j6})/\mu_3 + (t_{i4} t_{j4} + t_{i5} t_{j5})/\mu_4.$$

For matrix (13) to be positively determined, the necessary and sufficient condition is the positive values of the eigen moduli: $\mu_2 > 0$, $\mu_3 > 0$, and $\mu_4 > 0$. With allowance for (14), we obtain inequalities providing positive determinacy of matrix (13):

$$g_1 = 2(A_{31}/A_{11})^2 - 1 < A_{21}/A_{11} < 1, \qquad -1 < A_{31}/A_{11} < 1; \tag{15}$$

$$g_2 = (a/A_{11})(4a/A_{11} - 3) < b/A_{11} < a/A_{11}, \qquad 0 < a/A_{11} < 1. \tag{16}$$

The ranges of admissible values of the parameters of Eqs. (15) and (16) are shown in Figs. 1 and 2, respectively. An isotropic material is described by the lines $A_{21} = A_{31}$ (Fig. 1) and $b = 0$ (Fig. 2).

The inverse matrix a_{ij} for (13) can be obtained directly (without using eigen moduli and states):

$$a_{ij} = \begin{bmatrix} (A_{11}^2 - A_{31}^2)/A & & & & & \\ -(A_{11}A_{21} - A_{31}^2)/A & a_{11} & & & \text{sym} & \\ -A_{31}(A_{11} - A_{21})/A & a_{31} & (A_{11}^2 - A_{21}^2)/A & & & \\ 0 & 0 & 0 & (A_{11} - A_{31})^{-1} & & \\ 0 & 0 & 0 & 0 & a_{44} & \\ 0 & 0 & 0 & 0 & 0 & (A_{11} - A_{21})^{-1} \end{bmatrix}, \tag{17}$$

$$A = (A_{11} - A_{21})[A_{11}(A_{11} + A_{21}) - 2A_{31}^2] > 0.$$

Obviously, matrix (17) corresponds to a transversely isotropic material, but the matrix structure does not coincide with (13).

For material (13), matrices (3) and (6) acquire the form

$$L = \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ L_{21} & L_{22} & L_{32} \\ L_{31} & L_{32} & L_{33} \end{bmatrix},$$

$$L_{11} = (\lambda + 2\mu)\partial_{11} + (\mu - 2b/3)\partial_{22} + (\mu + b/3)\partial_{33} - \rho\partial_{44},$$

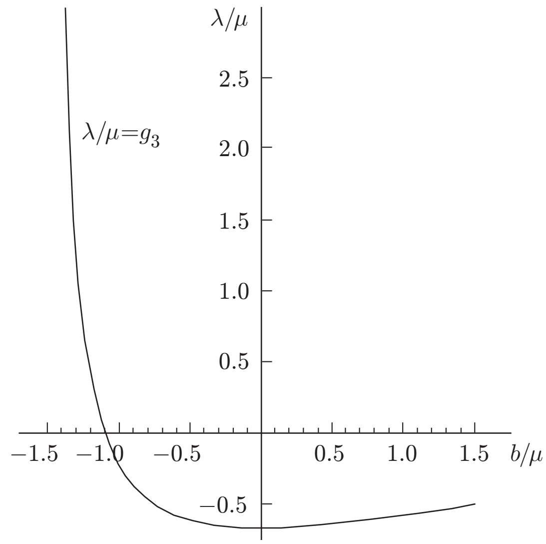

Fig. 3

$$L_{21} = (\lambda + \mu + 2b/3)\partial_{21}, \qquad L_{22} = (\mu - 2b/3)\partial_{11} + (\lambda + 2\mu)\partial_{22} + (\mu + b/3)\partial_{33} - \rho\partial_{44},$$

$$L_{31} = (\lambda + \mu - b/3)\partial_{31}, \qquad L_{32} = (\lambda + \mu - b/3)\partial_{32}, \tag{18}$$

$$L_{33} = (\mu + b/3)(\partial_{11} + \partial_{22}) + (\lambda + 2\mu)\partial_{33} - \rho\partial_{44},$$

$$T = \begin{bmatrix} \partial_1 & -\partial_2 & -\partial_{13} \\ \partial_2 & \partial_1 & -\partial_{23} \\ \partial_3 & 0 & \partial_{11} + \partial_{22} \end{bmatrix},$$

where

$$A_{11} = \lambda + 2\mu, \qquad a - b = \mu - 2b/3, \qquad a = \mu + b/3,$$
$$A_{11} - (a - b) = \lambda + \mu + 2b/3, \qquad A_{11} - a = \lambda + \mu - b/3. \tag{19}$$

Operators (18) obey the relation $LT = TD$, where $D = \operatorname{diag}(D_1, D_2, D_3)$ is determined by formulas (7).

The phase velocities (8) with allowance for (19) are

$$\rho|v|_1^2 = \lambda + 2\mu, \qquad \rho|v|_2^2 = \mu - 2b/3 + bn_3^2, \qquad \rho|v|_3^2 = \mu + b/3.$$

It follows from here that two velocities are independent of the direction of the wave normal n_k and one velocity depends on the component n_3. The extreme values of the velocity $\rho|v|_2^2$ are $a - b = \mu - 2b/3$ and $a = \mu + b/3$. If $b \leqslant 0$, then we have $0 < a \leqslant \rho|v|_2^2 \leqslant a - b < A_{11}$. If $b \geqslant 0$, then $0 < a - b \leqslant \rho|v|_2^2 \leqslant a < A_{11}$. The conditions of positive determinacy of (15) and (16) expressed via the parameters λ, μ, and b have the form

$$g_3 = \frac{2}{9}\frac{b}{\mu} - 1 + \frac{1}{2b/\mu + 3} < \frac{\lambda}{\mu}, \qquad -\frac{3}{2} < \frac{b}{\mu} < \frac{3}{2}. \tag{20}$$

The range of admissible values of the parameters of Eq. (20) is the set of all points above the curve $\lambda/\mu = g_3$ in Fig. 3. The constant λ can be negative. For an isotropic material, we have $b = 0$ and $-2/3 < \lambda/\mu$.

With allowance for (7), (18), and (19), we write the generic solution (4) of Eqs. (3):

$$u_1 = \partial_1\phi_1 - \partial_2\phi_2 - \partial_{13}\phi_3, \quad u_2 = \partial_2\phi_1 + \partial_1\phi_2 - \partial_{23}\phi_3, \quad u_3 = \partial_3\phi_1 + (\partial_{11} + \partial_{22})\phi_3; \tag{21}$$

$$[(\lambda + 2\mu)\partial_{kk} - \rho\partial_{44}]\phi_1 = f_1,$$
$$[(\mu - 2b/3)\partial_{kk} + b\partial_{33} - \rho\partial_{44}]\phi_2 = f_2, \quad [(\mu + b/3)\partial_{kk} - \rho\partial_{44}]\phi_3 = f_3; \tag{22}$$

$$\partial_1 f_1 - \partial_2 f_2 - \partial_{13} f_3 = 0, \quad \partial_2 f_1 + \partial_1 f_2 - \partial_{23} f_3 = 0, \quad \partial_3 f_1 + (\partial_{11} + \partial_{22}) f_3 = 0. \tag{23}$$

In turn, the generic solution of system (23) can be represented as

$$f_1 = (\partial_{11} + \partial_{22})(\partial_1\psi_1 + \partial_2\psi_2) + \partial_3\psi_3, \quad f_2 = \partial_{kk}(-\partial_2\psi_1 + \partial_1\psi_2), \quad f_3 = -\partial_3(\partial_1\psi_1 + \partial_2\psi_2) + \psi_3,$$

$$\partial_{kk}(\partial_{11} + \partial_{22})\psi_1 = 0, \qquad \partial_{kk}(\partial_{11} + \partial_{22})\psi_2 = 0, \qquad \partial_{kk}\psi_3 = 0.$$

A particular case of solutions (21), (22) for an isotropic material with the functions f_1, f_2, and f_3 being neglected is described in [13].

We consider the case of planar deformation for which $u_3 = 0$ and $\partial_3 = 0$. Then, instead of (18) and (21)–(23), we obtain

$$L = \begin{bmatrix} (\lambda + 2\mu)\partial_{11} + (\mu - 2b/3)\partial_{22} - \rho\partial_{44} & (\lambda + \mu + 2b/3)\partial_{12} \\ (\lambda + \mu + 2b/3)\partial_{21} & (\mu - 2b/3)\partial_{11} + (\lambda + 2\mu)\partial_{22} - \rho\partial_{44} \end{bmatrix},$$

$$T = \begin{bmatrix} \partial_1 & -\partial_2 \\ \partial_2 & \partial_1 \end{bmatrix}, \qquad LT = TD;$$

$$u_1 = \partial_1\phi_1 - \partial_2\phi_2, \qquad u_2 = \partial_2\phi_1 + \partial_1\phi_2,$$

$$[(\lambda + 2\mu)(\partial_{11} + \partial_{22}) - \rho\partial_{44}]\phi_1 = f_1, \qquad [(\mu - 2b/3)(\partial_{11} + \partial_{22}) - \rho\partial_{44}]\phi_2 = f_2, \tag{24}$$

$$\partial_1 f_1 - \partial_2 f_2 = 0, \qquad \partial_2 f_1 + \partial_1 f_2 = 0.$$

The following formulas are known [14]:

$$\partial_z = (1/2)(\partial_1 - i\partial_2), \qquad \partial_{\bar z} = (1/2)(\partial_1 + i\partial_2), \qquad z = x_1 + ix_2, \qquad i = \sqrt{-1}. \tag{25}$$

The last two equations of (24) are the Cauchy–Riemann conditions for the analytical function $\varphi_1'(z) = f_1 + if_2$ (the prime denotes the derivative with respect to z):

$$\partial_{\bar z}\varphi_1'(z) = (1/2)(\partial_1 + i\partial_2)(f_1 + if_2) = (1/2)(\partial_1 f_1 - \partial_2 f_2) + (i/2)(\partial_2 f_1 + \partial_1 f_2) = 0.$$

With allowance for (25), formulas (24) are written as

$$u_1 + iu_2 = 2\partial_{\bar z}(\phi_1 + i\phi_2); \tag{26}$$

$$2[(\lambda + 2\mu)(\partial_{11} + \partial_{22}) - \rho\partial_{44}]\phi_1 = \varphi_1'(z) + \overline{\varphi_1'(z)},$$

$$2i[(\mu - 2b/3)(\partial_{11} + \partial_{22}) - \rho\partial_{44}]\phi_2 = \varphi_1'(z) - \overline{\varphi_1'(z)}.$$

Under static conditions, we have $\partial_4 = 0$; since $\partial_{11} + \partial_{22} = 4\partial_z\partial_{\bar z}$, we obtain

$$2(\lambda + 2\mu)4\partial_z\partial_{\bar z}\phi_1 = \varphi_1'(z) + \overline{\varphi_1'(z)}, \qquad 2i(\mu - 2b/3)4\partial_z\partial_{\bar z}\phi_2 = \varphi_1'(z) - \overline{\varphi_1'(z)}.$$

The last relations yield

$$2(\phi_1 + i\phi_2) = [\bar z\varphi_1(z) + z\overline{\varphi_1(z)} + \psi_1(z) + \overline{\psi_1(z)}]/[4(\lambda + 2\mu)] + [\bar z\varphi_1(z) - z\overline{\varphi_1(z)} + \psi_2(z) - \overline{\psi_2(z)}]/[4(\mu - 2b/3)].$$

Here $\psi_1(z)$ and $\psi_2(z)$ are new analytical functions, which appeared due to integration. Using formula (26), we find

$$u_1 + iu_2 = 2\partial_{\bar z}(\phi_1 + i\phi_2) = \frac{\lambda + 3\mu - 2b/3}{4(\lambda + 2\mu)(\mu - 2b/3)}\,\varphi_1(z)$$

$$- \frac{\lambda + \mu + 2b/3}{4(\lambda + 2\mu)(\mu - 2b/3)}\,z\overline{\varphi_1'(z)} + \frac{1}{4(\lambda + 2\mu)}\,\overline{\psi_1'(z)} - \frac{1}{4(\mu - 2b/3)}\,\overline{\psi_2'(z)}. \tag{27}$$

We denote

$$\frac{\lambda + \mu + 2b/3}{4(\lambda + 2\mu)(\mu - 2b/3)}\,\varphi_1(z) = \varphi(z), \qquad \frac{1}{4(\lambda + 2\mu)}\,\overline{\psi_1'(z)} - \frac{1}{4(\mu - 2b/3)}\,\overline{\psi_2'(z)} = -\overline{\psi(z)},$$

then Eq. (27) takes the form

$$u_1 + iu_2 = \text{æ}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{28}$$

where

$$\text{æ} = \frac{3A_{11} - A_{21}}{A_{11} + A_{21}} = \frac{A_{11} + a - b}{A_{11} - (a - b)} = \frac{3(\lambda + 3\mu) - 2b}{3(\lambda + \mu) + 2b}.$$

Expression (28) corresponds to the Kolosov–Muskhelishvili formula [14] and, for $b = 0$, is a representation of displacements for an isotropic material; thereby, $æ = (\lambda + 3\mu)/(\lambda + \mu) = 3 - 4\nu$ (ν is Poisson's ratio). Thus, representation (28) of displacements in terms of analytical functions is valid for a transversely isotropic material corresponding to matrix (13) in the case of planar deformation. Therefore, all methods of functions of the complex variable, developed for an isotropic material [14], are also applicable in this case. In addition, solution (24) can be directly used in considering boundary-value problems and the generic representation (21)–(23) can be used in spatial problems.

There exist real elastic media close to a material with matrix (13) of elasticity moduli; for instance, for some ceramic materials [15], the parameter α [13] is roughly equal to unity:

$$\alpha = \frac{A_{44}/2 + A_{31}}{A_{11} - A_{44}/2} = \frac{A_{33} - A_{44}/2}{A_{44}/2 + A_{31}} \approx 1.$$

Note also, if the matrix a_{ij} of compliance coefficients of an anisotropic material has the structure of the form (5), (11), (13), then Young's modulus $1/E_n = n_i n_j a_{ijkl} n_k n_l$ in the direction n_i is independent of n_i and is identical for all directions: $1/E_n = a_{11}$, as in an isotropic material (see also [4]).

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